

Diffusion of charged particles in a stochastic force-free magnetic field

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ABSTRACT

We consider a random stationary magnetic field with zero average magnetic field, $\langle B \rangle = 0$. In the case when carriers of electric current, that creates a field, are electrons the field is force-free, $\text{curl} \mathbf{B} = \alpha \mathbf{B}$. In a small region, in which the coefficient of α and the strength $|B|$ can be considered constants, the force-free field is the vector rotating in the direction perpendicular to the plane in which magnetic field lines lie. The motion of a charged particle in such a field is described by a mathematical pendulum and continuously traces the transition from magnetized motion, when the cyclotron radius r_c less than the characteristic scale of inhomogeneity $L = 2\pi/\alpha$ (trapped particles), to almost free, when $r_c > L$ (untrapped particles). Averaging over the magnetic field spectrum and value of α gives the diffusion coefficient of particles D depending on the cyclotron radius of a particle motion in the large-scale field r_L and its correlation length L_0 . The diffusion increases proportionally to the first power of the Larmor radius, $D \propto r_L$, at $r_L < L_0/2\pi$, and to the square of the Larmor radius, $D \propto r_L^2$, for $r_L > L_0/2\pi$. For the cosmic rays in the Galaxy, in which there are wide ranges of large-scale field strength B_{LS} and the correlation length L_0 , averaging over these values gives the dependence of the diffusion coefficient as the fractional power of the Larmor radius r_m in the maximum large-scale field, $D \propto r_m^{(1-\sigma)/(1+\beta)}$. The value of β is the index of the spectrum of the large-scale field, $B_{LS} \propto L_0^\beta$, and $1 - \sigma$ is the index in the distribution function $f(L_0)$ over scales, $f(L_0) \propto L_0^{-1+\sigma}$. For the Kolmogorov spectrum of the magnetic field, $\beta = 1/3$, and almost flat spectrum over scales, $\sigma = 1/15$, the value of the index $(1 - \sigma)/(1 + \beta)$ is 0.7, which corresponds to observed dependence of the diffusion of cosmic rays in the Galaxy over their energy.

1 INTRODUCTION

Motion of charged particles in a stochastic magnetic field for large times can be described by the diffusion equation. The most important problem is to calculate the coefficient of the spatial diffusion of particles D . Depending on the energy particles can be divided into magnetized, whose Larmor radius r_c is much smaller than the correlation length of the magnetic field L , and the non-magnetized. For the first approximation non-magnetized particles move along almost straight lines, directions of which change slowly. Calculating the diffusion coefficient of non-magnetized particles one can use the approximation of the delta-correlated in time magnetic field. This problem has been solved, see, for example, Plotnikov, Pelletier & Lemoine (2011). To describe the motion of magnetized particles is much more difficult. Magnetized particles are divided into two types: trapped, many fold reflecting by magnetic mirrors, and untrapped. Most developed now the approach of calculation of diffusion coefficients of magnetized particles is the quasilinear theory (QLT), firstly proposed by Jokipii (1966). In the QLT it is assumed that the particle trajectory coincides with the line of the average magnetic field B , and deviations from that by magnetic field fluctuations are small. It allows to find the pitch angle

diffusion coefficient and estimate the mean free path length of a particle. However, this approach is strictly applicable only in case of small fluctuations, $\delta B \ll B$. In recent years, different authors have attempted to generalize the QLT. For example, in the paper (Shalchi et al., 2009) there was proposed the QLT of the second order. The value of a fluctuating magnetic field was considered as a small parameter similar to in the QLT. However, the particle trajectories was calculated up to the first order in the sum of the average and the perturbed magnetic field. As a result, the obtained expression for the D is of the second order of the magnitude of fluctuations. Next, the results were extended without changes for the case of large fluctuations, $\delta B \gg B$. If in the standard QLT for $\delta B \ll B$ the integral over time contains the delta function, in the QLT of the second order it is replaced by a Gaussian function with the relative width of $\delta B/(B\mu)$, where μ is the cosine of the pitch angle. We see that in the limit $\delta B \gg B$ we take place with the function of a very large width. This transition from the delta-function to the function of a very large width is necessary to recognize at least unfounded.

Another generalization of the QLT is Nonlinear Guiding Center theory (NLGC), proposed by Matthaeus et al. (2003). In this theory two key assumptions used. The first is the assumption that the pair correlation function of particle velocities, $\langle v(t)v(t') \rangle$, is

known, and it depends only on the difference of times $t-t'$. Second, the strong assumption about the spatial distribution of the particles is made, namely, the average value of $\langle e^{i\vec{k}\vec{r}(t)} \rangle$ is written by hand. This makes it possible to obtain an integral equation for the value of D_\perp which is the diffusion coefficient across the mean magnetic field. It should be noted that NLGC, as well as many other QLT, constructed generally on the assumption that particles move along the magnetic field lines. In fact, it solves the problem of random walk the magnetic field lines. However, in reality, along the lines move only untrapped particles. As soon as the particle moves in a trapped state, its motion is not therefore well described.

The main question that is explored in the aforementioned articles is the anisotropy of diffusion coefficients D_\perp/D_\parallel . Often it is assumed that the dependence of D_\parallel on the particle energy is already known. Another approach for calculation of D is used by Dogiel et al. (1987). In this paper it is assumed that on the scale less than the correlation length of the magnetic field L a magnetized particle moves along a straight line. That gives the estimate of coefficient of diffusion $D = vL$. However, this assumption is rather rough. This approach also takes into account only untrapped particles, and can give too high value for the diffusion coefficient.

Some astrophysical applications use the Bohm diffusion approximation (Bohm & Burhop, 1949). See, for example, Zirakashvili & Ptuskin (2012). The Bohm diffusion, $D_B = cT/16eB = v_T^2/32\omega_c$, is the anomalous diffusion of particles across the magnetic field B , related to their scattering by a plasma turbulence. In this case the characteristic time of the particle scattering, τ , is of order of the period of the cyclotron rotation in the magnetic field, $\tau = 2^5/3\omega_c$, rather than the time between classical collisions. For the Bohm diffusion, with respect to the collisionless diffusion in a random magnetic field, we have $D \approx v r_L$.

An alternative way of calculating of the D is a numerical calculation, see Casse, Lemoine & Pelletier (2002); Snodin et al. (2016). The magnetic field $\mathbf{B}(\mathbf{r}) = \mathbf{B} + \delta\mathbf{B}(\mathbf{r})$ is realized on the discrete grid in the space. Fluctuations $\delta\mathbf{B}(\mathbf{r})$ are generated as the sum of plane waves with random phases, polarizations and directions. They corresponds to a fixed spectrum over wave vectors. The spectrum is usually selected as the Kolmogorov one. Thereafter equations of motion of particles in this magnetic field are solved. The result is averaged over a large number of particles, and magnetic field realizations. Authors observe output on to the diffusion mode after long time, and measure the diffusion coefficient. In the paper by Casse, Lemoine & Pelletier (2002) the diffusion coefficient for magnetized particles at an arbitrary ratio between δB and B was calculated for the Kolmogorov spectrum, and result of QLT was confirmed: $D \propto r_L^{1/3}$. However, in the recent article (Snodin et al., 2016) the following approximation of the numerical results was obtained

$$D_\parallel = 0.75v r_L + \frac{1}{3}vL \frac{B^2}{\delta B^2} \left(\frac{r_L}{L} \right)^{1/3}. \quad (1)$$

At zero average field this expression gives the Bohm diffusion coefficient $D \propto r_L$. It should be noted that in the process of generation of a magnetic field the range of wave vectors, over which the summation occurs, usually is no more than $k_{max}/k_{min} = 256$. This limits the range of the ratio r_L/L by two orders of magnitude. Perhaps this is the reason in discrepancies of results of numerical calculations by different groups. In any case, the analytical approach provides answers regardless of numerical schemes and parameters used, and has a great importance for understanding the physics of diffusion process in a random magnetic field.

2 PARTICLE DIFFUSION

Let a charged particle having a charge of q and a velocity of v_0 , moving in a stationary magnetic field $\mathbf{B}(\mathbf{r})$. We introduce the notations: B_0 , so that $B_0^2 = \langle \mathbf{B}^2 \rangle$, and $\mathbf{b} = \mathbf{B}/B_0$. We also introduce the cyclotron frequency of a particle in the magnetic field of the magnitude B_0 , $\omega_c = qB_0/mc\gamma$, where $\gamma = (1 - v_0^2/c^2)^{-1/2}$. Random magnetic field $\mathbf{B}(\mathbf{r})$ has no mean value, $\langle \mathbf{B} \rangle = \mathbf{0}$, and its pair correlation function is $\langle B_i(\mathbf{r})B_j(\mathbf{r} + \mathbf{R}) \rangle = \Phi_{ij}(\mathbf{R})$. It is a smooth differentiable function of \mathbf{R} , decreasing at large distances R . The structure of the correlator $\Phi_{ij}(\mathbf{R})$ investigated in works related to the study of the magnetic dynamo in a turbulent conducting medium (Schekochihin, Boldyrev & Kulsrud, 2002), as well as in turbulent weakly ionized gas, from which molecular clouds consist of (Istomin & Kiselev, 2013). In this paper we describe the correlator $\Phi_{ij}(\mathbf{R})$ by one parameter, it is the correlation length L_0 of a random magnetic field.

The particle motion is determined by the system of equations

$$\begin{aligned} \frac{dr_i}{dt} &= v_i, \\ \frac{dv_i}{dt} &= \omega_c e_{ikl} v_k b_l. \end{aligned} \quad (2)$$

Here e_{ikl} is the unit antisymmetric tensor. We assume that the motion of particles becomes the diffusive one for sufficiently long time, i.e. $\langle r^2(t) \rangle = 2Dt$. Here averaging $\langle \dots \rangle$ denotes averaging over realization of a random magnetic field. Consequently,

$$\frac{d\langle r^2 \rangle}{dt} = 2\langle v_i r_i \rangle = 2D = \text{const}(t).$$

Differentiating this relation with respect to time, and considering $\langle v_i v_i \rangle = v_0^2$, we obtain

$$e_{ikl} \langle r_i v_k b_l \rangle = -\frac{v_0^2}{\omega_c}.$$

Again, by differentiating with respect to time and using a condition of stationarity of magnetic field, $db_i/dt = v_m \partial b_i / \partial r_m$, we arrive to the relation

$$e_{ikl} \langle r_i \frac{dv_k}{dt} b_l \rangle + e_{ikl} \langle r_i v_k v_m \frac{\partial b_l}{\partial r_m} \rangle = 0.$$

Substituting the acceleration dv_k/dt from the equations of motion (2), we obtain

$$\omega_c e_{ikl} e_{kmn} \langle r_i v_m b_n b_l \rangle + e_{ikl} \langle r_i v_k v_m \frac{\partial b_l}{\partial r_m} \rangle = 0 \quad (3)$$

Let suppose that the fourth-order correlation functions can be splitted into pairs: $\langle r_i v_m b_n b_l \rangle = \langle r_i v_m \rangle \langle b_n b_l \rangle$ and $\langle r_i v_k v_m \partial b_l / \partial r_m \rangle = \langle v_k v_m \rangle \langle r_i \partial b_l / \partial r_m \rangle$. Assuming that the magnetic field and the velocity field is completely isotropic, $\langle b_n b_l \rangle = \delta_{nl}/3$, $\langle v_k v_m \rangle = v_0^2 \delta_{km}/3$, we get

$$2\omega_c \langle r_i v_i \rangle = v_0^2 e_{ikl} \langle r_i \frac{\partial b_l}{\partial r_k} \rangle. \quad (4)$$

Using the relation $D = \langle v_i r_i \rangle$ we obtain for the diffusion coefficient

$$D = \frac{v_0^2}{2\omega_c} \langle \mathbf{r} \text{curl } \mathbf{b} \rangle. \quad (5)$$

3 FORCE-FREE FIELD

Thus, we see that the diffusion of particles is determined by the electric current $\mathbf{j} \propto \text{curl } \mathbf{b}$ creating the magnetic field. In turn,

the electric current in a plasma is the electron current. Then, in the absence of an electric field, the Ampere force acting on the electrons must be equal to zero because of the electron mass is small, $\mathbf{jB} = 0$. The magnetic field in this case is force-free, $\text{curl } \mathbf{B} = \alpha \mathbf{B}$. The value of $\alpha(\mathbf{r})$ is constant along the magnetic field line, $\nabla_\alpha \mathbf{B} = 0$, and can change across. To imagine how looks a force-free random magnetic field and what is the motion of a charged particle in it, we consider small local region where the magnetic field gradient in a direction perpendicular to the field can be considered constant. Select the coordinate z along this direction. The local magnetic field then has the following configuration

$$\mathbf{B}(\mathbf{r}) = B_0(\sin \alpha z, \cos \alpha z, 0). \quad (6)$$

This configuration satisfies the condition $\text{curl } \mathbf{B} = \alpha \mathbf{B}$, $\alpha = \text{const}$. Equations of motion in the field, specified by that, have the form

$$\frac{d\mathbf{v}}{dt} = \omega_c \begin{pmatrix} -v_z \cos \alpha z \\ v_z \sin \alpha z \\ v_x \cos \alpha z - v_y \sin \alpha z \end{pmatrix}$$

The first two equations are integrated

$$\begin{aligned} v_x + \frac{\omega_c}{\alpha} \sin \alpha z &= C_1 \\ v_y + \frac{\omega_c}{\alpha} \cos \alpha z &= C_2 \end{aligned}$$

This allows us to get rid of velocity components v_x, v_y and to reduce this system to one equation for v_z ,

$$\frac{dv_z}{dt} = -C\omega_c \sin(\alpha z - \theta_0), \quad (7)$$

where we have introduced the notations $C = (C_1^2 + C_2^2)^{1/2}$ and θ_0 , defined by the equation $C_1/C = \sin \theta_0$, $C_2/C = \cos \theta_0$. Let introduce the new variable

$$\psi = \alpha z - \theta_0, \quad (8)$$

and we get the equation of the mathematical pendulum

$$\frac{d^2\psi}{dt^2} = -\alpha C\omega_c \sin \psi. \quad (9)$$

Integrating this equation we obtain the energy conservation law

$$\frac{v_z^2}{2} - \frac{C\omega_c}{\alpha} \cos \psi = \text{const} \stackrel{\text{def}}{=} E.$$

Introduce the notation $\kappa^2 = 2C\omega_c/(\alpha E + C\omega_c)$, and get

$$\frac{d\psi}{dt} = \pm \frac{2}{\kappa} (\alpha C\omega_c)^{1/2} \left(1 - \kappa^2 \sin^2 \frac{\psi}{2}\right)^{1/2}. \quad (10)$$

We see that charged particles in a force-free field are divided into two classes: trapped in the direction of z , for which $\kappa > 1$, and untrapped in the direction of z , $\kappa < 1$. The trapped particles oscillate between the points $\psi = \pm 2\arcsin(1/\kappa)$. The characteristic amplitude of speed oscillations in the z -direction is $\delta v_z \simeq 2(\omega_c/\alpha)^{1/2}|v_\parallel|$. Here v_\parallel is the velocity of the particle in the plane (x, y) , along the direction of the magnetic field. Note that oscillations of the amplitude of ω_c/α are superimposed on the free longitudinal motion of particles. In the strong magnetic field, $\omega_c > \alpha v_0$, fluctuations of velocity along z of trapped particles is small, $\delta v_z < \omega_c/\alpha$, which corresponds to the fact that the cyclotron radius, $\delta v_z/\omega_c < 1/\alpha$, is less than the characteristic scale of the magnetic field change. Trapped particles correspond to particles executing a cyclotron rotation in a constant magnetic field. In an inhomogeneous magnetic field not all particles execute a finite

motion, some of them, oscillating along the direction of z , move freely along this direction. The period of the particle motion is not equal to the period of the cyclotron rotation, $2\pi/\omega_c$. As for particles located on the separatrix $\kappa = 1$, the period is equal to infinity. This is the main influence on the inhomogeneity on the particle motion.

To calculate the diffusion coefficient, we need to find a product of $\mathbf{r} \text{curl } \mathbf{b}$. For the force-free field $\text{curl } \mathbf{b} = \alpha \mathbf{b}$, so we calculate the value of $\mathbf{r}\mathbf{b} = x b_x + y b_y$. Let write

$$x(t) = x_0 + \int_0^t dt' v_x = x_0 + (v_{x0} + \frac{\omega_c}{\alpha} \sin \alpha z_0)t - \frac{\omega_c}{\alpha} \int_0^t \sin(\alpha z(t')) dt'. \quad (11)$$

A similar expression we have for $y(t)$. Expression (11) contains the initial coordinates of the particles x_0, z_0, v_{x0} . Because the diffusive motion does not depend on initial coordinates, which are equally probable, we perform averaging over the initial coordinates, $\overline{x_0} = \overline{v_{x0}} = \overline{\sin(\alpha z_0)} = 0$. Thus, the expression for $\mathbf{r}\mathbf{b}$ reduces to

$$\overline{\mathbf{r}(\mathbf{t})\mathbf{b}(\mathbf{r}(t))} = -\frac{\omega_c}{\alpha} \int_0^t dt' \cos(\alpha(z(t') - z(t)))$$

Turn to the integration over ψ ,

$$\overline{\mathbf{r}\mathbf{b}} = \pm \frac{\kappa}{2\alpha} \left(\frac{\omega_c}{\alpha C}\right)^{1/2} \int_{\psi_0}^{\psi} d\psi' \frac{\cos(\psi' - \psi)}{(1 - \kappa^2 \sin^2 \frac{\psi'}{2})^{1/2}}.$$

The integral is calculated in terms of elliptic functions

$$\begin{aligned} \frac{1}{2} \int_{\psi_0}^{\psi} d\psi' \frac{\cos(\psi' - \psi)}{(1 - \kappa^2 \sin^2 \frac{\psi'}{2})^{1/2}} &= \cos \psi \left[\frac{2}{\kappa^2} (E(\psi/2, \kappa) - \right. \\ &\left. F(\psi/2, \kappa)) + F(\psi/2, \kappa) \right] - \frac{2 \sin \psi}{\kappa^2} (1 - \kappa^2 \sin^2 \frac{\psi}{2})^{1/2}. \end{aligned}$$

The configuration of the force-free magnetic field in the form of (6) is a unit cell in the picture of the magnetic field, repeating through the period $2\pi/\alpha$ over coordinate z . We assume a random force-free magnetic field as a random superposition of such unit cells having a random orientation and a random value of α . Therefore, during the passage of a particle over one cell we have $\psi - \psi_0 = 2\pi$. Substituting these limits into the previous integral, and taking into account that

$$E(\frac{\psi_0}{2} + \pi, \kappa) - E(\frac{\psi_0}{2}, \kappa) = 2E(\kappa);$$

$$F(\frac{\psi_0}{2} + \pi, \kappa) - F(\frac{\psi_0}{2}, \kappa) = 2K(\kappa),$$

we get

$$\overline{\mathbf{r}\mathbf{b}} = \frac{2\kappa}{\alpha} \left(\frac{\omega_c}{\alpha C}\right)^{1/2} \cos \psi_0 \left[\frac{2}{\kappa^2} (E(\kappa) - K(\kappa)) + K(\kappa) \right].$$

Here $K(\kappa)$ and $E(\kappa)$ are complete elliptic integrals of the first and second kinds, respectively. The obtained expression is valid for untrapped particles, $\kappa < 1$. For trapped particles, $\kappa > 1$, it should be analytically continued into the region $\kappa > 1$. That corresponds to the integration over ψ between the turning points $\sin(\psi/2) = \pm 1/\kappa$. Remind that particles are trapped only in z direction (cyclotron rotation), but are free in directions x and y . As a result, we obtain

$$\overline{\mathbf{r}\mathbf{b}} = \frac{2}{\alpha} \left(\frac{\omega_c}{\alpha C}\right)^{1/2} \cos \psi_0 G(\kappa),$$

where

$$G(\kappa) = \begin{cases} 2(E(\kappa) - K(\kappa))/\kappa + \kappa K(\kappa), & \kappa < 1 \\ 2E(1/\kappa) - K(1/\kappa), & \kappa > 1 \end{cases} \quad (12)$$

Let introduce the ratio of the particle Larmor radius v_0/ω_c to the

characteristic scale of change of the magnetic field, which in this case is the value of $1/\alpha$,

$$a = \frac{\alpha v_0}{\omega_c}. \quad (13)$$

We obtain

$$\overline{D} = \frac{2v_0^2}{\omega_c} F(a),$$

where

$$F(a) = \left(\frac{v_0}{aC}\right)^{1/2} \cos \psi_0 G(\kappa).$$

We now average over initial velocities, considering the magnitude of the velocity is constant, $|v| = v_0$. Suppose that at the initial time $t = 0$ the coordinate $z = z_0$ and the velocity components are given in the form

$$v_{x0} = v_0 \sin \phi \sin \delta, \quad v_{y0} = v_0 \sin \phi \cos \delta, \quad v_{z0} = v_0 \cos \phi.$$

Let denote $v = C/v_0$, $\tilde{z}_0 = \alpha z_0$ and express all quantities through ϕ, δ and \tilde{z}_0 ,

$$v^2 = \sin^2 \phi + 2a^{-1} \sin \phi \cos(\tilde{z}_0 - \delta) + a^{-2},$$

$$\cos \psi_0 = \cos(\tilde{z}_0 - \theta_0) = \frac{1}{v} (\sin \phi \cos(\tilde{z}_0 - \delta) + a^{-1}),$$

$$\kappa^2 = \frac{4v}{a \cos^2 \phi + 2v(1 - \cos \psi_0)}.$$

Replacing of z_0 on \tilde{z}_0 allows us to see that $F(a)$ does not depend on α , except through a . Average of $F(a)$ over ϕ, δ uniformly on the sphere and with the uniform distribution of \tilde{z}_0 inside the interval $0 < \tilde{z}_0 < 2\pi$ is

$$\overline{F(a)} = \frac{1}{8\pi^2} \int_0^{2\pi} d\tilde{z}_0 \int_0^\pi \sin \phi d\phi \int_0^{2\pi} F(a) d\delta.$$

The integral is calculated numerically for different values of the parameter a . The value of $\overline{F(a)}$ can be accurately approximated by the function

$$\overline{F(a)} = \begin{cases} F_0 - F_1 a, & a < a_0 \\ F_2 a^{-\chi}, & a > a_0. \end{cases} \quad (14)$$

The fit gives these settings: $a_0 = 1.0$, $\chi = 1.143$, $F_0 = 1.59 \simeq \pi/2$, $F_1 = 0.33$, $F_2 = 0.94$. However, if use only the tail of the fitted function $\overline{F(a)}$ with $a > 3$, we get

$$\overline{F(a)} = 0.80 a^{-1.008}. \quad (15)$$

Thus, if $a \gg 1$ the function $\overline{F(a)} \propto a^{-1}$ with good accuracy. The exact value of $\overline{F(a)}$ and the results of the piecewise approximation are presented in Figure 1. Power form of $\overline{F(a)}$ at $a > 1$ is confirmed by calculations presented in Figure 2.

3.1 Averaging over the spectrum

After averaging over initial velocities we have the expression for the diffusion coefficient only through the parameters of magnetic field α and B_0

$$D(\alpha, B_0) = \frac{2v_0^2}{\omega_c} \overline{F(a)},$$

where, let remind, $\omega_c = qB_0/mc\gamma$, $a = \alpha v_0/\omega_c$. Now we average of D over the spectrum of magnetic fluctuations $B_0(r)$. We choose

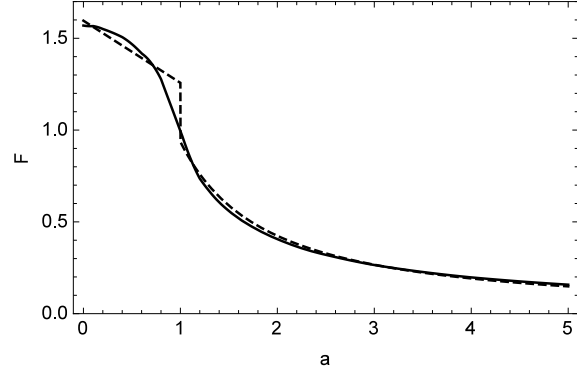


Figure 1. The graph of $\overline{F(a)}$ (solid line) and the results of the piecewise approximation (dashed line).

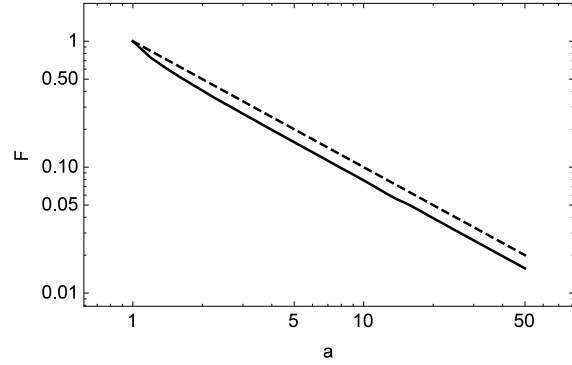


Figure 2. The dependence of $\overline{F(a)}$ on a at $a > 1$. Coordinates are in the logarithmic scale. The dotted line shows the straight line a^{-1} .

it as a power law

$$B_0(r) = B_{LS} \left(\frac{r}{L_0} \right)^\beta, \quad (16)$$

where L_0 is the correlation length of the magnetic field, B_{LS} is the magnitude of large-scale magnetic field, β is the index of the spectrum. For the Kolmogorov spectrum $\beta = 1/3$. Averaging over the spectrum has the form

$$\langle D \rangle = \frac{1}{L_0} \int_0^{L_0} dr D \left[\alpha = \frac{2\pi}{r}, B_0 = B_0(r) \right]. \quad (17)$$

Introduce the notations

$$\omega_0 = \frac{eB_{LS}}{mc\gamma}, \quad r_L = \frac{v_0}{\omega_0}$$

for the cyclotron frequency ω_0 and the Larmor radius r_L in the large-scale magnetic field B_{LS} . We also introduce the parameter

$$A = \frac{2\pi r_L}{L_0}.$$

Then, the averaging over the spectrum is given by

$$\langle D \rangle = \frac{2v_0^2}{L_0} \int_0^{L_0} dr \frac{\overline{F(a(r))}}{\omega_c(r)}.$$

For the chosen spectrum (16) $\omega_c(r) = \omega_0(r/L_0)^\beta$, $a(r) = A(r/L_0)^{-\beta-1}$, and we have

$$\langle D(A) \rangle = \frac{v_0 A}{\pi} \int_0^{L_0} \overline{F(a)} \left(\frac{r}{L_0} \right)^{-\beta} dr. \quad (18)$$

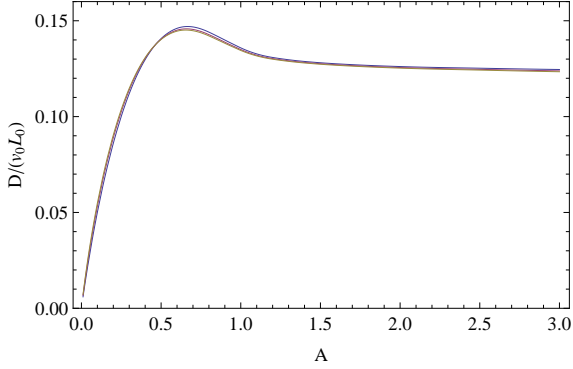


Figure 3. Numerical calculations of $D(A)$ for $\beta = 0.2$, $\beta = 1/3$, $\beta = 0.4$.

Replacing the integration variable r to a , we finally obtain

$$\langle D(A) \rangle = \frac{v_0 L_0}{\pi(\beta + 1)} \int_A^\infty \overline{F(a)} \left(\frac{A}{a} \right)^\delta da, \quad (19)$$

where the exponent is $\delta = 2/(\beta + 1)$. It can be seen that the result depends only on the parameter A .

Substitute into the integral (19) the result of the piecewise approximation of the function (14). The integral was calculated analytically. Given that $\chi - 1 = 0.14 \ll 1$, we obtain

$$\frac{\langle D(A) \rangle}{v_0 L_0} = \begin{cases} \frac{A}{\pi(\beta+1)} \left(\frac{F_0}{\delta-1} (1-A^{\delta-1}) - \frac{F_1}{2-\delta} (A^{\delta-1} - A) + \frac{F_2}{\delta} A^{\delta-1} \right), & A < 1 \\ \frac{F_2}{2\pi} A^{1-\chi}, & A > 1 \end{cases} \quad (20)$$

If $A \ll 1$, the first term of the expansion over A has the form

$$\langle D(A) \rangle = \frac{F_0}{\pi(1-\beta)} v_0 L_0 A = \frac{2F_0}{(1-\beta)} v_0 r_L = \frac{\pi}{(1-\beta)} v_0 r_L,$$

giving the Bohm diffusion coefficient $D \propto v_0 r_L$.

If $A > 1$

$$\langle D(A) \rangle = \frac{F_2}{2\pi} v_0 L_0 A^{-0.008} \approx \frac{F_2}{2\pi} v_0 L_0.$$

To check the formula (20), we numerically calculate the integral (19), using previously counted function $\overline{F(a)}$, for different values of the index β . From the Figure 3 we see that at $A > 1$ the result does not depend on β , as follows from (20). Figure 4 shows a comparison of the results of the numerical calculation with that given by the formula (20) for $\beta = 0.33$. We see a good agreement that allows us to continue using the simple formula (20) for the diffusion coefficient. The fit of the numerical results with $A > 2$ gives $D \propto A^{-0.07} \approx \text{const.}$

3.2 Large Larmor radius

In the previous section we have shown that for large Larmor radius $D \propto \text{const.}$ This means that our calculations describes only the diffusion of particles with $r_L < L_0$ and does not apply to particles with $r_L > L_0$. However, the diffusion coefficient of the particle with $r_L \gg L_0$ is well known. See, for example, Plotnikov, Pelletier & Lemoine (2011). Passing only one region of the size of $\simeq L_0$, particle is only slightly deviates from the nearly rectilinear motion by a small angle $\delta\phi$. Only when the particle passes a long way $s \gg L_0$ its motion becomes diffusive. Then the diffusion coefficient D is equal to $v_0^2 \tau / 3$. Here the time τ is equal to the turn of

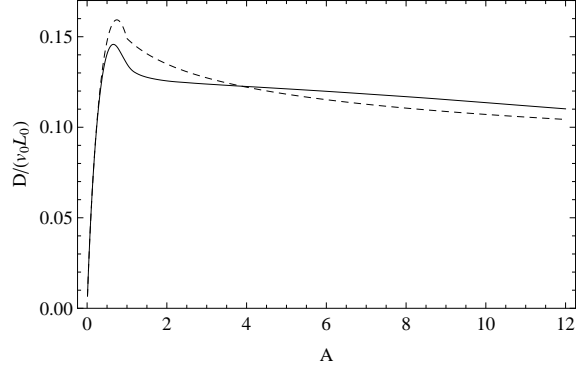


Figure 4. Numerical (solid line) and analytical (dashed line) calculations of $D(A)$, $\beta = 0.33$.

the particle, $\phi \simeq 2\pi$. Therefore, $2D_\phi \tau = (2\pi)^2$, where the value of D_ϕ is the diffusion coefficient of scattering over the angle ϕ , $D_\phi = d\langle \delta\phi^2 \rangle / 2dt$.

We define the change of the angle as $(\delta\phi)^2 = \delta v_i^2 / v_0^2$. To find the diffusion over the angle, we use the equations of motion of the particle (2). Let write

$$\delta v_i(t) = \frac{q}{mc\gamma} \int_0^t dt' e_{ikl} v_k B_l. \quad (21)$$

From that we have

$$\frac{1}{2} \frac{d}{dt} \langle \delta\phi^2 \rangle = \left(\frac{q}{mc\gamma v_0} \right)^2 \int_0^t dt' e_{ikl} e_{imn} \langle v_k(t') B_l(t') v_m(t) B_n(t) \rangle \quad (22)$$

Assuming that the velocity of the particle deviates little from the original one, $v_i \approx v_{0i}$, and using the relation $\langle v_k v_m \rangle = v_0^2 \delta_{km} / 3$, we obtain

$$\frac{1}{2} \frac{d}{dt} \langle \delta\phi^2 \rangle = \frac{2}{3v_0} \left(\frac{q}{mc\gamma} \right)^2 \int_0^{L_0} dr \langle B_i(0) B_i(r) \rangle \quad (23)$$

We define the pair correlation function of the magnetic field in the form

$$\langle B_i(0) B_i(r) \rangle = B_{LS}^2 \left[1 - \left(\frac{r}{L_0} \right)^{2\beta} \right]. \quad (24)$$

Finally we find

$$D_\phi = \frac{4\beta}{3(2\beta+1)} \frac{v_0 L_0}{r_L^2}. \quad (25)$$

Thus, if we assume that the structure of a random magnetic field is a superposition of regions having the same correlation length L_0 and the same intensity of large-scale magnetic field B_{LS} , then at $r_L > L_0/2\pi$ the diffusion coefficient is

$$\langle D \rangle = \frac{\pi^2 (2\beta+1)}{2\beta} \frac{v_0 r_L^2}{L_0}. \quad (26)$$

Summarizing the previous result for the case of $r_L < L_0/2\pi$, the diffusion coefficient at $\beta = 1/3$ can be represented approximately as

$$\langle D \rangle \simeq \begin{cases} \frac{3\pi}{2} v_0 r_L, & r_L < L_0/2\pi \\ 3\pi^2 \frac{v_0 r_L^2}{L_0}, & r_L > L_0/2\pi. \end{cases} \quad (27)$$

The numerical coefficient in the limit of large Larmor radius was received only as estimation. Therefore, we slightly changed the numerical coefficient so that $\langle D \rangle$ becomes continuous at $r_L = L_0/2\pi$. We see that the diffusion of particles is mainly determined

by their motion in the large-scale magnetic field and weakly depends on the spectrum of magnetic fluctuations. Index β of the spectrum (16) appears only as coefficient before the dependence on r_L and L_0 . Since the Larmor radius of the particle is proportional to its rigidity, $r_L = \mathcal{R}/qB_{LS}$, then for small rigidities the diffusion coefficient is directly proportional to the rigidity (energy), $D \propto \mathcal{R}$, then passing to quadratic dependence, $D \propto \mathcal{R}^2$, at higher rigidity. The resulting dependence (27) has a simple physical interpretation. Diffusion of particles is determined by the particle motion in the large-scale field (its amplitude is maximal for spectral indices $\beta > 0$), and increases with the Larmor radius.

4 DIFFUSION OF COSMIC RAYS IN THE GALAXY

The structure of the magnetic field in the Galaxy is much more complicated than is represented by the expression (16). First, large-scale magnetic field varies in a wide range, $10^{-6} G \leq B_{LS} \leq 10^{-4} G$. Second, the size of L_0 also varies widely, from distance $\sim 10^{12} cm$ to the size of $\sim 100 pc$, which is the size of large clouds of ionized warm and neutral cold gases. It is therefore necessary to average particle motion over different regions having different values of the correlation length L_0 and magnitudes of large-scale field B_{LS} . We assume the distribution function $f(L_0)$ over scales is a power law decreasing with increasing L_0 (it is normalized to unity),

$$f(L_0) = \sigma L_0^{-1} \left(\frac{L_0}{L_m} \right)^\sigma, \quad \sigma < 1.$$

For small values of parameter $\sigma \ll 1$ this spectrum is close to the flat: the relative number of objects of $n(0 < L_0 < L_1)$ in the range $0 < L_0 < L_1$ equals $n = (L_1/L_m)^\sigma \simeq const(L_1)$. Choosing a distribution over amplitudes of the magnetic field it is naturally to continue the dependence (16) onto large scales,

$$B_{LS} = B_m \left(\frac{L_0}{L_m} \right)^\beta. \quad (28)$$

Here L_m is the maximum scale, B_m is the maximum amplitude of the magnetic field. In regions with different amplitudes of the magnetic field the particle has the different Larmor radius. The distribution over the Larmor radii corresponds to the distribution over the magnetic field, $r_L = r_m(L_0/L_m)^{-\beta}$, where the value of r_m is equal to the Larmor radius of the particle in the field B_m , $r_m = v_0 mc\gamma/qB_m$. We introduce the parameter of magnetization of the particle in the magnetic field of the maximum scale,

$$\rho = 2\pi \frac{r_m}{L_m}. \quad (29)$$

Let us first consider a particle with $\rho < 1$, i.e. magnetized in the field of maximum strength. Averaging of the diffusion coefficient over different scales is divided into two regions, $L_0 > L_{cr}$ and $L_0 < L_{cr}$, where

$$L_{cr} = L_m \rho^{\frac{1}{1+\beta}}. \quad (30)$$

In the region $L_0 > L_{cr}$ particles are magnetized, and $D(L_0) = 3\pi v_0 r_L/2 = 3\pi v_0 r_m (L_m/L_0)^\beta/2$. Averaging over scales $L_{cr} < L_0 < L_m$ gives

$$D_1 = \int_{L_{cr}}^{L_m} dL_0 f(L_0) D(L_0) = \frac{3\pi\sigma}{2} v_0 r_m \int_{L_{cr}}^{L_m} dL_0 \frac{L_0^{\sigma-1}}{L_m^\sigma} \left(\frac{L_m}{L_0} \right)^\beta = \frac{3\sigma}{4(\beta-\sigma)} v_0 L_m \left(\rho^{\frac{1+\sigma}{1+\beta}} - \rho \right).$$

At $L_0 < L_{cr}$ particles are not magnetized, and the diffusion coefficient is determined by the diffusion over the angle,

$$D_2 = \frac{2\pi^2}{3} \frac{v_0^2}{\langle D_\phi \rangle}, \quad (31)$$

to which we substitute the value of D_ϕ averaged over different scales L_0

$$\langle D_\phi \rangle = \int_0^{L_{cr}} dL_0 f(L_0) D_\phi(L_0) = \int_0^{L_{cr}} dL_0 \frac{\sigma L_0^{\sigma-1}}{L_m^\sigma} \frac{4\beta}{3(2\beta+1)} \frac{v_0 L_0}{r_m^2} \left(\frac{L_0}{L_m} \right)^{2\beta} = \frac{16\pi^2}{3} \frac{\sigma\beta}{(2\beta+1)(2\beta+\sigma+1)} \frac{v_0}{L_m} \rho^{\frac{\sigma-1}{1+\beta}}.$$

Then

$$D_2 = \frac{(2\beta+\sigma+1)(2\beta+1)}{8\sigma\beta} v_0 L_m \rho^{\frac{1-\sigma}{1+\beta}}. \quad (32)$$

Thus, the average diffusion coefficient for the particles with $\rho < 1$ is

$$\langle D \rangle = D_1 + D_2 = \frac{(2\beta+\sigma+1)(2\beta+1)}{8\sigma\beta} v_0 L_m \left\{ \rho^{\frac{1-\sigma}{1+\beta}} + \frac{6\sigma^2\beta}{(2\beta+\sigma+1)(2\beta+1)(\beta-\sigma)} \left[\rho^{\frac{1+\sigma}{1+\beta}} - \rho \right] \right\}. \quad (33)$$

For small values of the index σ the main term in this sum is the first one, which defines the dependence of the diffusion coefficient on the rigidity, $D \propto \mathcal{R}^{(1-\sigma)/(1+\beta)}$. Index $(1-\sigma)/(1+\beta)$ is less than 1, $(1-\sigma)/(1+\beta) < 3/4$ for $\beta = 1/3$. So, for the Kolmogorov spectrum, $\beta = 1/3$, and $\sigma = 1/15$ the value of the index is 0.7, which corresponds to observations (Berezinskii et al., 1990). Figure 5 shows the dependence $\langle D(\rho) \rangle$ (33) for $\beta = 1/3$, $\sigma = 1/15$. It is evident that practically there is no difference from the law $\langle D \rangle \propto r_m^{(1-\sigma)/(1+\beta)}$. Thus, the main contribution to the spatial diffusion of charged particles with $\rho < 1$ gives regions of relatively small sizes, $L_0 < L_{cr}$, in which particles are not magnetized. The contribution to the diffusion of large scale regions, $L_0 > L_{cr}$, in which particles spend large time, at least $6\sigma^2\beta/(2\beta+\sigma+1)(2\beta+1)(\beta-\sigma) \simeq 10^{-2}$ times less. At $L_m \simeq 100 pc$ and $B_m \simeq 10^{-4} G$ for particles of energy of $\simeq 10 GeV$ the diffusion coefficient (33) for $\sigma = 1/3$ and $\beta = 1/15$ is of the order of $1.5 \cdot 10^{28} cm^2/s$, which corresponds to observations (Berezinskii et al., 1990).

In the region of large Larmor radius, $r_m > L_m/2\pi$, the diffusion coefficient begins to increase proportionally to the square of the rigidity (energy)

$$\langle D \rangle = \frac{(2\beta+\sigma+1)(2\beta+1)}{8\pi\sigma} v_0 L_m \rho^2.$$

However, at $L_m \simeq 100 pc$ and $B_m \simeq 10^{-4} G$, this corresponds to energies of protons $\mathcal{E} > 10^{16} eV$. Particles of such high energies are already poorly trapped in the disk of the Galaxy and fill the halo.

5 DISCUSSION

We have considered the motion of charged particles in a random force-free magnetic field, $curl\mathbf{B} = \alpha\mathbf{B}$. Such random field can be represented as a superposition of the individual areas in which the field changes along the direction which is perpendicular to the field, while remaining constant in absolute value. The magnetic field vector rotates in a plane in which it lies, while the coordinate perpendicular to this plane is changing. The characteristic scale of variation of the field is $L = 2\pi/\alpha$. The important thing is that the

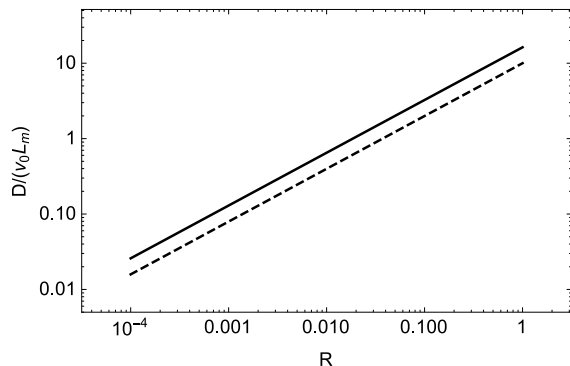


Figure 5. Dependence of the diffusion coefficient $\langle D \rangle / v_0 L_m$ (33) on $\rho = 2\pi r_m / L_m$ for $\beta = 1/3$, $\sigma = 1/15$ (solid line). The dashed line is $10\rho^{0.7}$, its slope factually does not differ from that given by the expression (33)

motion of a particle in this field ($\alpha \simeq \text{const}$) can be accurately described, even in the case when the cyclotron radius of the particle r_c is equal to or of the order of the size of the field inhomogeneity $2\pi/\alpha$. We know the particle trajectory in an arbitrary field only for two cases: 1) when $r_c \ll L$ and 2) when $r_c \gg L$. In the first case the particle moves along the magnetic field, making cyclotron rotation, and drifting in the transverse direction under the influence of the gradient field strength (gradient drift) and the curvature of the magnetic field lines (curvature drift). In the second limiting case a particle moves almost along a straight line, slightly deflecting in the direction perpendicular to the magnetic field. In the case of a force-free field a particle performs nonlinear oscillations in the direction perpendicular to the plane in which a magnetic field lies. Part of particles performs the finite motion (trapped particles). It is the analogue of the cyclotron rotation. However, the period of their motion is not equal to the period of the cyclotron rotation, $2\pi/\omega_c$, although the magnetic field strength remains constant. They equal only when $r_c \ll 2\pi/\alpha$, the period increases when the cyclotron radius approaches to the length of the field inhomogeneity. On the separatrix, $\kappa = 1$, (see the equation (10) the period of motion is equal to infinity. Next, for $\kappa < 1$ the particle starts to move freely, when passing through the separatrix, oscillating in the direction of the field gradient (untrapped particles). It traced a continuous transition from magnetized the free motion.

Furthermore, averaging over the initial coordinates and velocities of particles, considering them as equally probable, we find the contribution of particle motion in a single cell of the random field into the diffusion coefficient. It turns out that the answer depends only on the dimensionless parameter, $a = \alpha v_0 / \omega_c$, which is the ratio of the cyclotron radius to the length of the inhomogeneity. For small values of a , i.e. small cyclotron radii (magnetized particles), the contribution to the diffusion is maximum and begins to decrease at large values of a . If $a \gg 1$ the contribution to the diffusion of these particles tends to zero according to the law $\propto a^{-1}$. This is natural, since the free motion can not be diffusion.

Since the configuration of the random field contains a continuous range of different values of the magnetic field strength and scale, we take averaging over their values, considering spectrum as power law (16). As a result, it turns out, that the diffusion of particles is mainly determined by the cyclotron radius of particle rotation r_L in the large-scale field and its scale L_0 . The index of the spectrum β enters only as multiplier into the expression for the diffusion coefficient. Diffusion is proportional to the Larmor radius of

the particle, r_L , at $r_L < L_0/2\pi$ for all reasonable spectral indices β , and is independent of the cyclotron radius at $r_L > L_0/2\pi$. However, in this case we do not take into account that particle continues to move in the magnetic fields at large distances $r > L_0$ and it has to path a long way, scattering on many irregularities, in order its motion becomes the diffusion. Calculating the diffusion coefficient over angle of deflection from rectilinear motion D_ϕ (25) for the case $r_L > L_0/2\pi$, we find that the spatial diffusion coefficient for particles with large cyclotron radius is proportional to the square of its cyclotron radius. Thus, the diffusion coefficient of particles in a random magnetic field, representing a superposition of regions having the same value of the large-scale field B_{LS} and the same values of scales L_0 , has the form represented by the formula (27). It is linear, then quadratic dependence on the cyclotron radius in large-scale field.

As for the motion of cosmic rays in the Galaxy, since values of B_{LS} and L_0 have here wide ranges of parameters we must average the motion of a particle over these different areas. We assumed that the distribution over strengths of large-scale field has the same power law dependence on sizes as well as for small-scale field (16). As for the distribution over scales of large-scale field, we also consider it as the power law, $f(L_0) \propto L_0^{-1+\sigma}$, $\sigma < 1$. After averaging we found that the diffusion coefficient depends on the particle cyclotron radius by the formula $D \propto r_L^{(1-\sigma)/(1+\beta)}$. For values of the quantities $\beta = 1/3$ (the Kolmogorov spectrum) and $\sigma = 1/15$ (almost flat Harrison-Zel'dovich spectrum, Harrison (1970), Zel'dovich (1972)) we obtain the spectral index 0.7, which coincide with that observed in the Galaxy for cosmic rays.

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